J - homomorphism $\quad J: \pi_r SO \longrightarrow \pi_r^S$

- Relation w/ EHP sequence

$$\pi_r SO(n) \longrightarrow \pi_r H(n) = \pi_r \Omega^n S^n = \pi_{n+r} S^n.$$

- Relation w/ $|\Theta_n|$. # of smooth str. on S^n. Kervaire - Milnor thy.
- What Adams did ?

Strategy of pf.

I. Lower Bound of $|\mathrm{im}\, J|$.

<u>Recall</u> $\tilde{K}$ = cpx K - thy

Cohomology operations = nat. trans. from $\tilde{K}$ to $\tilde{K}$.

<u>Thm</u> $\exists!$ cohomlgy operations on $\tilde{K}$. $\psi^k : \tilde{K} \Longrightarrow \tilde{K}$. $k \geq 0$

s.t. 1) $\forall f: X \to Y$. $\psi^k f^* = f^* \psi^k$

2) L line bundle. $\psi^k(L) = L^k$

3) $\psi^k \circ \psi^\ell = \psi^{k\ell} = \psi^\ell \circ \psi^k$. $k, \ell \geq 0$.

4) $\psi^k(x) = k^m x$. $x \in \tilde{K}(S^{2m}) = \mathbb{Z}$. $m, k \geq 0$

5) $\psi^p(a) \equiv a^p \mod p$. p prime.

X cpt Hausdorff. ψ^k ring homphism.

- <u>Construction</u> $\lambda_t : \mathrm{Vect}_{\mathbb{C}}(X) \longrightarrow \tilde{K}(X)[[t]]$

$$E \longmapsto \sum_{k \geq 0} \Lambda^k(E)\, t^k$$

exterior alg.

factors through
$$\begin{tikzcd} \mathrm{Vect}_{\mathbb{C}}(X) \arrow[r, "\lambda_t"] \arrow[d, "\text{gp completion}"'] & \tilde{K}(X)[[t]] \\ \tilde{K}(X) \arrow[ur, dashed, "\exists!\ \tilde{\lambda}_t"'] & \end{tikzcd}$$

Set
$$\psi_t : \tilde{K}(X) \longrightarrow \tilde{K}(X)[[t]]$$
$$E \longmapsto \psi^0(E) - t \frac{d}{dt} \ln \lambda_{-t}(E)$$

$\psi^0(E)$ = trivial bundle $/X$, rk $=$ rk E

$$\psi_t(E) = \sum_{k\geq 0} \psi^k(E) t^k$$

$\psi^k(E)$: Adams operation.

e.g. Want $\psi^k(L) = L^k$

$$\begin{aligned} \psi_t(L) &= \sum_{k\geq 0} \psi^k(L) t^k \\ &= 1 - t\frac{d}{dt} \ln \lambda_{-t}(L) \\ &= 1 - t\frac{d}{dt} \ln \sum_{k\geq 0} \Lambda^k(L)\cdot(-t)^k \\ &= 1 - t\frac{d}{dt} \ln(1 - Lt). \\ &= 1 - \frac{t}{1-Lt}\cdot(-L) \\ &= \frac{1}{1-Lt} = \sum_{k\geq 0} (Lt)^k = \sum_{k\geq 0} L^k t^k \end{aligned}$$

- Chern character.

$\exists$ nat. trans. $ch : \tilde{K} \Rightarrow H^{\text{even}}(-;\mathbb{Q})$ s.t.

1) $ch(X)$ ring homo.

2) $\forall$ L line bundle. $ch(L) = e^{c_1(L)}$, $c_1 = 1^{st}$ Chern class

3) $m \geq 0$, $ch(S^{2m}) : \tilde{K}(S^{2m}) \longrightarrow H^{\text{even}}(S^{2m};\mathbb{Q})$

iso onto subgp $H^{even}(S^{2m} ; \mathbb{Q})$

subgp $= H^{even}(S^{2m} ; \mathbb{Z})$.

Construction $\forall$ rk k cpx v.b. $\xi : E \to X$

$ch(E) := \sum_{r \geq 0} \frac{1}{r!} s_r(E)$ $\quad$ s_r = r^{th} Newton poly evaluated at $c(E)$

$$s_r(E) := S_r(c_1(E), \dots, c_k(E))$$

$$S_r \in \mathbb{Z}[\sigma_1, \dots, \sigma_k]$$

$\sigma_i \in \mathbb{Z}[x_1, \dots, x_k]$ i^{th} elementary sym. poly

$\sigma_1 = x_1$, $\sigma_2 = x_1 x_2 + x_2 x_1$

$$S_r(\sigma_1, \dots, \sigma_k) = x_1^r + \dots + x_k^r.$$

- e - invariant

$X \xrightarrow{f} Y \longrightarrow C_f = Y \cup_f CX$ $\quad$ cofib seq.

$\longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y$

$f^* = 0$, $(\Sigma f)^* = 0$ in $\tilde{K}$.

$\Rightarrow 0 \leftarrow \tilde{K}(Y) \leftarrow \tilde{K}(C_f) \longleftarrow \tilde{K}(\Sigma X) \longleftarrow 0.$

Def $\quad$ d - invariant of f. $d(f) = f^*$

e - invariant of f. $e(f) \in \mathrm{Ext}^1(\tilde{K}(Y), \tilde{K}(\Sigma X))$

- $f \sim g$. $e(f) = e(g)$

$e(f+g) = e(f) + e(g)$ $\quad$ [Adams J(X) - IV § 3.]

Rk d. & e : Toda bracket $\longrightarrow$ Massey products.

So far. e-inv. + Chern character + Adams operation.

Consider $f: S^{2(k+n)-1} \longrightarrow S^{2k}$. $f^* = 0$

$\Rightarrow \quad 0 \leftarrow \tilde{K}(S^{2k}) \leftarrow \tilde{K}(C_f) \leftarrow \tilde{K}(S^{2(k+n)}) \leftarrow 0$

$f \in \pi_{2(k+n)-1} S^{2k}$. can ignore k. $k=0$.

$$\begin{tikzcd} 0 & \tilde{K}(S^0) \arrow[l] \arrow[d, "ch"'] & \tilde{K}(C_f) \arrow[l] \arrow[d, "ch"'] & \tilde{K}(S^{2n}) \arrow[l] \arrow[d, "ch"'] & 0 \arrow[l] \\ 0 & H^{even}(S^0; \mathbb{Q}) \arrow[l] & H^{even}(C_f; \mathbb{Q}) \arrow[l] & H^{even}(S^{2n}; \mathbb{Q}) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

$\forall\ a \in \tilde{K}(C_f)$. $a \longmapsto 1 \in \tilde{K}(S^0)$

$$ch(a) = (1, \tilde{e})$$

$$\in H^0(S^0) \oplus H^{2n}(S^{2n})$$

$$1 \in H^0(S^0). \quad \tilde{e} \in H^{2n}(S^{2n}).$$

$$e(f) = \tilde{e} \,/\, \mathrm{im}\left(\tilde{K}(S^{2n}) \longrightarrow H^{even}(S^{2n})\right) \in \mathbb{Q}/\mathbb{Z}.$$

Goal: $e(f)$

Aside $\quad \tilde{K}(X) = [X, BU \times \mathbb{Z}]$

$\widetilde{KO}(X) = [X, BO \times \mathbb{Z}]$

$X = S^m \Rightarrow \quad \pi_{m-1} U \overset{\text{l.e.s.}}{=} \pi_m BU = \tilde{K}(S^m) \longrightarrow \pi_{m-1} S^0$

$\pi_{m-1} O \overset{\text{l.e.s.}}{=} \pi_m BO = \widetilde{KO}(S^m) \longrightarrow \pi_{m-1} S^0$

via $U \longrightarrow O$

factorization

$$\begin{tikzcd} \pi_{m-1} U \arrow[r] \arrow[d] & \pi_{m-1} S^0 \\ \pi_{m-1} O \arrow[ur] & \end{tikzcd}$$

Bott $\Rightarrow$

π_*	0	1	2	3	4	5	6	7
U	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
O	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$.

FACT $\quad \pi_{m-1} U \to \pi_{m-1} O$

iso $\quad m-1 \equiv 3 \mod 8$

$\cdot 2 \quad m-1 \equiv 7 \mod 8$.

Work at $\pi_{2m-1} U \longrightarrow \pi_{2m-1} S^0$

$$\begin{array}{c} \pi_{2m-1} U \\ \| \\ \tilde{K}(S^{2m}) \\ \downarrow ch \\ H^{even}(S^{2m}; \mathbb{Q}) \end{array}$$

Natural to consider $\pi_{2m-1} U \xrightarrow{J} \pi_{2m-1} S^0 \xrightarrow{e} \mathbb{Q}/\mathbb{Z}$

Take $x_{2m} \in \pi_{2m} BU \rightsquigarrow x_{2m} \in \tilde{K}(S^{2m})$ regarded as cpx v.b.

$x_{2m} : E \longrightarrow S^{2m}$

$$f = J(x_{2m}) \in \pi_{2m-1} S^0$$

Consider $e(f) \in \mathbb{Q}/\mathbb{Z}$

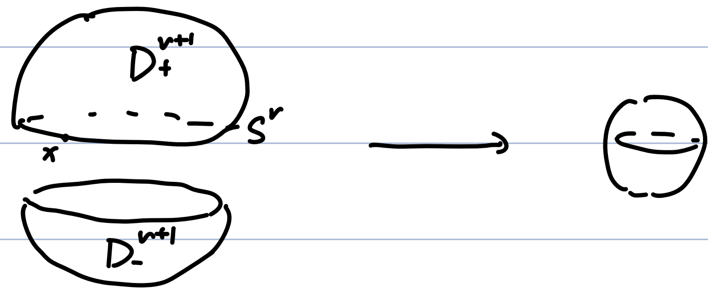
Need to know $\tilde{K}(C_f) = \tilde{K}(S^0 \cup_f e^{2n})$.

Thm $\quad \xi : E \longrightarrow S^{r+1}$ v.b. $\quad \varphi : S^r \longrightarrow U(n)$

i.e. $E = (D_+^{r+1} \times \mathbb{C}^n \sqcup D_-^{r+1} \times \mathbb{C}^n)/\sim$

$(x, v) \sim (x, \varphi(x) v) \quad \forall x \in$ equator

attach via $\varphi : x \mapsto \underset{U(n)}{\underset{\cap}{\varphi(x)}}$

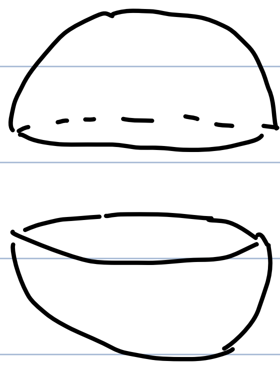


$v \in$ fiber $\longmapsto \varphi(x) v.$

Then $Th(\xi) = S^q \cup_{J\varphi} C(S^{q+r})$ for some q

$= S^q \cup_{J\varphi} D^{q+r+1}$

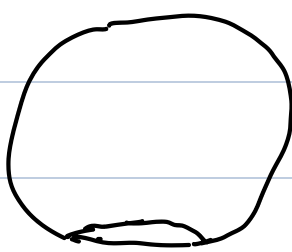
pf Sketch.



D_+^{n+1} fib : $D^q \cong \mathbb{C}^q$

$\longleftarrow \varphi : S^n \longrightarrow U(n)$

D_-^{r+1} fib : $D^q \cong \mathbb{C}^q$

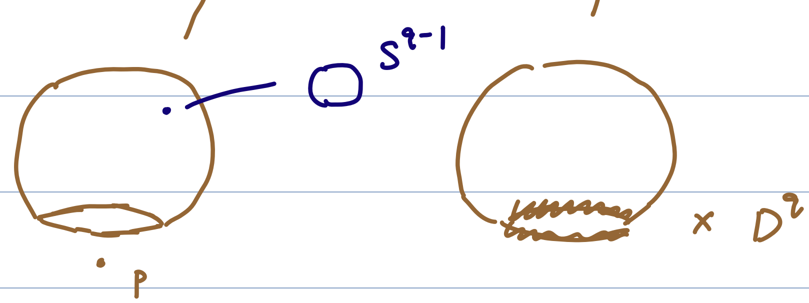


D_+^{r+1} fib D^q

• pt $\quad p \quad$ fib D^q

$\Rightarrow E = (D^{n+1} \times D^q) \cup (D^q \times \{p\}).$

$S^{q+r} \simeq \partial(D^q \times D^{n+1}) \simeq (\partial D^q \times D^{n+1}) \times (D^q \times \partial D^{n+1})$



$Th(\xi)$ = glue two parts via φ.

= apply φ to $\partial(D^q \times D^{r+1}) \simeq S^{q+r}$. collapse

rest of S^{q+r}. as J did.

[Eva Belmont thesis § 3.2].
UCSD.

- $x_{2n} \in \tilde{K}(S^{2n})$ regarded as cpx v.b.

$$x_{2n} : E \longrightarrow S^{2n}$$

$$f = J(x_{2n}) \in \pi_{2n-1} S^0$$

$$\tilde{K}(C_f) = \tilde{K}(S^0 \cup_f e^{2n}) \overset{Th}{=} \tilde{K}(Th(x_{2n})) \qquad q=0, \ r=2n-1$$

$$J\varphi = f . \ x_{2n} = \varphi.$$

Now Thom iso $E^i X \xrightarrow{\cong} E^{i+n}(Th(\xi))$. $\xi : E \to X$ rk n

$$x \longmapsto x \cup u , \quad u \in E^{i+n}(Th(\xi)).$$

u = Thom class in $\tilde{K}$. $\quad ch : \tilde{K}(C_f) \longrightarrow H^{even}(C_f ; \mathbb{Q})$

$$\begin{array}{ccc} \tilde{K}(C_f) & \longrightarrow & H^{even}(C_f ; \mathbb{Q}) \\ Th \ \| & & \| \ Th. \\ \tilde{K}(Th(x_{2n})) & & H^{even}(Th(x_{2n}) ; \mathbb{Q}) \\ u & \longmapsto & ch(u) \end{array}$$

Write u_H rational Thom class in H^{even}

$$ch(u) = u_H \cdot \mathcal{X}(x_{2n}) , \quad \mathcal{X}(x_{2n}) \in H^{even}(S^{2n})^{\times}$$

is a unit.

Def $\mathcal{X}: \tilde{K}(X) \longrightarrow H^{\text{even}}(X)^{\times}$ for any v.b. $E \xrightarrow{\xi} X$ is called a "cannibalistic class". i.e. $\mathcal{X}(\xi) = \operatorname{ch}(u_\xi)/u_{H_\xi}$

Warning: not the same as Adams did!

Prop 1) $\mathcal{X}(\xi \oplus \eta) = \mathcal{X}(\xi)\,\mathcal{X}(\eta)$.

2) $\underline{n}$ = trivial n bundle / pt. $\mathcal{X}(\underline{n}) = 1$.

Goal $\mathcal{X}(L)$? L line bundle. by splitting principle.

Aside (splitting principle) $\xi: E \to X$ vk n. X cpt Haus.

$\exists$ $p: F(E) \longrightarrow X$. $F(E)$ flag bundle ass. E s.t.

$p^*: \tilde{K}(X) \longrightarrow \tilde{K}(F(E))$ inj. $p^*E = \bigoplus_{i=1}^{n} L_i$

$L_i \in \tilde{K}(F(E))$

$$\begin{tikzcd}
\bigoplus_{i=1}^{n} L_i = p^*E \arrow[r] \arrow[d] & E \arrow[d, "\xi"] \\
F(E) \arrow[r, "p"] & X
\end{tikzcd}$$

- Compute $\mathcal{X}(L)$.

Step 1: $Th(L \to \mathbb{CP}^\infty) \cong \mathbb{CP}^\infty$. L canonical line bundle.

pf Sketch: L_n canonical line bundle $\mathbb{CP}^n$

Claim $L_n \cong \mathbb{CP}^{n+1}$ $\xrightarrow{\lim}$ $L \cong \mathbb{CP}^\infty$

$Th(L \to \mathbb{CP}^\infty) = D(L)/S(L) \simeq D(L) \simeq L \cong \mathbb{CP}^\infty$.

$S(L) = S^\infty \simeq *$

$L_n \subseteq \mathbb{CP}^n \times \mathbb{C}^{n+1}$. Let $L_n \longrightarrow \mathbb{CP}^{n+1}$

$$(x, v) \longmapsto [\, v_i : x_0 : \cdots : x_n \,]$$

$i = 1^{st}$ index s.t. $x_i \neq 0$.

Step 2: $\tilde{K}(Th(L \to \mathbb{CP}^\infty)) \xrightarrow{\cong} \tilde{K}(\mathbb{CP}^\infty)$

$$u_L \longmapsto 1 - L$$

Note $ch(L) = e^{c_1(L)}$. Let $x \in H^2(S^2; \mathbb{Z})$, one has

$$ch(L) = e^{-x} \quad (\text{"$-$" sign convention \textcolor{red}{?}})$$

$$H^{even}(Th(L \to \mathbb{CP}^\infty)) \cong H^{even}(S^2)$$

$$u_H = x \longleftarrow\!\shortmid \; x$$

$\Rightarrow$ u_L Thom class of $\tilde{K}(Th(L \to \mathbb{CP}^\infty))$

$$ch(u_L) = ch(1-L) = 1 - ch(L) = 1 - e^{-x}$$

$$= \chi(L) \cdot x$$

$\Rightarrow$ $\chi(L) = \dfrac{1-e^{-x}}{x}$. $x = -c_1(L)$.

- Compute $\chi(x_{2n})$ $\quad x_{2n} \in \pi_{2n} BU$. $\quad x_{2n} : E \to S^{2n}$

$$\begin{tikzcd}
\bigoplus_{i=1}^{n} L_i \arrow[r] \arrow[d] & E \arrow[d, "x_{2n}"] \\
\mathbb{CP}^n \arrow[r] & S^{2n}
\end{tikzcd}$$

FACT

$\tilde{K}(S^2) = \tilde{K}(\mathbb{CP}^n)$

$= \dfrac{\mathbb{Z}[L]}{(1-L)^2}$.

$$Th(L_i \longrightarrow \mathbb{CP}^n) \rightsquigarrow 1 - L_i$$

$$Th\Big(\bigoplus_{i=1}^{n} L_i \longrightarrow \mathbb{CP}^n\Big) \rightsquigarrow \prod_{i=1}^{n} (1 - L_i)$$

Note $\chi(L) = \dfrac{1-e^{-x}}{x}$

$n=1$: x_2

$$\chi(x_2) = \chi(1-L) = \frac{1}{\chi(L)} = \frac{x}{1-e^{-x}}$$

$n=2$: x_4

$$\chi(x_4) = \chi((1-L_1)(1-L_2)) = \chi(1-L_1-L_2+L_1L_2)$$

$$= \frac{\chi(L_1)\,\chi(L_2)}{\chi(L_1L_2)}$$

$$c_1(L_1L_2) = c_1(L_1) + c_1(L_2).$$

Denote $g(x) = \frac{1-e^{-x}}{x}$ $\Rightarrow$ $\chi(x_4) = \frac{g(y_1+y_2)}{g(y_1)\,g(y_2)}$

$y_1 = c_1(L_1)$ $\quad = 1 + e\cdot y_1 y_2 + \cdots$

$y_2 = c_1(L_2)$

In general. $\chi(x_{2n}) = \frac{\prod \text{ even \# of sum of } g}{\prod \text{ odd \# of sum of } g}$

$$= 1 + e\cdot y_1 y_2 \cdots y_n + \cdots.$$

e is the desired thing.

Take log:

$$\log \chi(x_{2n}) = -(h(y_1) + h(y_2) + \cdots + h(y_n))$$

$$+ (h(y_1+y_2) + h(y_1+y_3) + \cdots + h(y_{n-1}+y_n))$$

$$+ \cdots$$

$$+ (-1)^{n+1} h(y_1 + \cdots + y_n)$$

$h(y) = \log g(y)$

<u>FACT</u> • $h_{k,1} = x_1^k$

$h_{k,2} = x_1^k + x_2^k - (x_1+x_2)^k$

$$h_{k,3} = x_1^k + x_2^k + x_3^k - (x_1+x_2)^k - (x_1+x_3)^k - (x_2+x_3)^k$$
$$+ (x_1+x_2+x_3)^k$$

$$\vdots$$

$$h_{k.r} = \begin{cases} k!\ x_1 \cdots x_k & , \quad k = r \\ 0 & , \quad \text{else.} \end{cases}$$

$$h(y) = \log g(y) = \log \frac{1-e^{-y}}{y}$$

$$\frac{d}{dy} h(y) = \frac{d}{dy} \log g(y) = \frac{d}{dy} \log \frac{1-e^{-y}}{y}$$
$$= \frac{1}{y}\left(\frac{y}{e^y - 1} - 1 \right)$$
$$= \frac{1}{y}\left(\sum_{k=0}^{\infty} B_k \frac{y^k}{k!} - 1 \right)$$
$$= \sum_{k=1}^{\infty} \frac{B_k}{k} \frac{y^{k-1}}{(k-1)!}$$

$$\Rightarrow \quad \log g(y) = \sum_{k=1}^{\infty} \frac{B_k}{k} \frac{y^k}{k!} = h(y)$$

$$\log \chi(x_{2n}) = -(h(y_1) + h(y_2) + \cdots + h(y_n))$$
$$+ (h(y_1+y_2) + h(y_1+y_3) + \cdots + h(y_{n-1}+y_n))$$
$$+ \cdots$$
$$+ (-1)^{n+1} h(y_1 + \cdots + y_n)$$
$$= \sum_{k=1}^{\infty} \frac{B_k}{k} \cdot \frac{1}{k!} h_{k,n} = \frac{B_n}{n} \cdot \frac{1}{n!} \cdot n!\ y_1 \cdots y_n$$
$$= \frac{B_n}{n} y_1 \cdots y_n .$$

$$\chi(x_{2n}) = 1 + e\, y_1 y_2 \cdots y_n + \cdots$$

$$\Rightarrow \quad e(Jx_{2n}) = \frac{B_n}{n}$$

- So far. $x_{2n} \in \pi_{2n} BU$

$$\begin{tikzcd} x_{2n} \in \pi_{2n} BU \arrow[r, "J"] \arrow[d, "\alpha"'] & \pi_{2n-1} S^0 \arrow[r, "e"] & \mathbb{Q}/\mathbb{Z} \\ \pi_{2n} BO \arrow[ur] & & \end{tikzcd}$$

w/ $e(J(x_{2n})) = \frac{B_n}{n}$ $\qquad e \rightsquigarrow \widetilde{K}$

α iso $\quad 2n \equiv 4 \mod 8$

$\cdot 2 \quad 2n \equiv 0 \mod 8$

$e_{\mathbb{R}} \rightsquigarrow \widetilde{KO}$. $e_{\mathbb{R}}(f) \in H^{even}(S^{2n}; \mathbb{Q}) / \widetilde{KO}(S^{2n})$.

$\Rightarrow \quad e_{\mathbb{R}}(J(x_{2n})) = \frac{B_{2n}}{4n} \qquad x_{2n} \in \pi_{4n} BO$.

Now

$$\begin{tikzcd} \pi_{4k} BO \arrow[r, "J"] & \pi_{4k-1} S^0 \arrow[r, "e_{\mathbb{R}}"] & \mathbb{Q}/\mathbb{Z} \\ x_{4k} \arrow[rr, mapsto] & & \frac{B_{2k}}{4k} \end{tikzcd}$$

$\Rightarrow$ $e_{\mathbb{R}} \circ J(x_{4k})$ factors through a cyclic gp of order of denominator of $\frac{B_{2k}}{4k}$

$$\Rightarrow \begin{tikzcd} \operatorname{im} J \arrow[r, "\varphi"] \arrow[d, hook, "\mathrm{incl}"'] & \mathbb{Z}/\text{denominator of } \frac{B_{2k}}{4k} \\ \pi_{4k-1} S^0 \arrow[ur, "e_{\mathbb{R}}"'] & \end{tikzcd}$$

$|\operatorname{im} J| \geq$ denominator of $\frac{B_{2k}}{4k}$.

II. Upper bound.

<u>Adams Conj</u>: $k \in \mathbb{N}$. $\forall x \in \widetilde{KO}(X)$. one has

$$k^n(\psi^k(x) - x) = 0 \quad \text{in } J(X) = \operatorname{im} J. \quad n >> 0.$$

<u>Claim</u> Adams Conj $\Rightarrow$ $|\operatorname{im} J| \le$ denominator of $\frac{B_{2k}}{4k}$.

pf Sketch. $J''(X) = \widetilde{KO}(X)/H$

$H = \bigcap_f H_f$. $H_f = \langle k^{f(k)}(\psi^k x - x) :$

$f: \mathbb{N} \to \mathbb{N} \rangle$

Want to show $J''(X) \longrightarrow J(X)$ surjective.

$J(X) \cong \operatorname{im} J$.

can compute $|J''(X)|$. $X = S^{4m}$.

$|J''(S^{4m})| = s(2m) =$ denominator of $\frac{B_{2m}}{4m}$.

Write $y \in \widetilde{K}(S^{4m})$. $k^{f(k)}(\psi^k(y) - y) = k^{f(k)}(k^{2m} - 1)y$.

H_f consists of multiple of $h(f, 2m)$ = highest common factor of the integers $k^{f(k)}(k^{2m} - 1)$. $k \in \mathbb{Z}$.

Note that $\widetilde{KO}(S^{4m}) = \mathbb{Z}$

$\widetilde{KO}(S^{4m}) \longrightarrow \widetilde{K}(S^{4m}) = \mathbb{Z}$ non-zero

Take $x \in \widetilde{KO}(S^{4m})$ replace y by x above

$\Rightarrow$ $|\operatorname{im} J| \le s(2m) =$ denominator of $\frac{B_{2m}}{4m}$. $\square$

[Adams $J(X)$ - II . § 3 ?].

<u>Conclude</u> e : inv. by "factoring through" denominator of $\mathbb{Q}/\mathbb{Z}$

$\Rightarrow$ $\operatorname{im} J \ge$ denominator of e

$\Rightarrow$ lower bound.

Adams Conj $\Rightarrow$ upper bound.

To compute the lower bound, we need to know

$$\begin{tikzcd}
\tilde{K}(C_f) \arrow[r, equal, "\text{Thm}"] \arrow[d, "ch"'] & \tilde{K}(Th(x_{2n})) \arrow[d, "ch"] \\
H^*(C_f) & H^*(Th(x_{2n}))
\end{tikzcd}$$

$x_{2n} \in \pi_{2n} BU$

gen $\longmapsto$ gen. Thom class

$\downarrow$

rational | Thom class

difference cannibalistic class

cannibalistic class $\Rightarrow$ result of e

- Quillen

$$\begin{tikzcd}
BGL(\mathbb{F}_q) \arrow[r, "\alpha"] & BU \arrow[r] \arrow[d, "\mu"'] & C \arrow[dl, "\beta"] \\
 & BF[p^{-1}] &
\end{tikzcd}$$

Brauer lifting

$C =$ mapping cone of α.

μ induced from $x \longmapsto J(\psi^k x - x)$

Quillen $\Rightarrow$ $\mu \cdot \beta$ null homotopic

$\Rightarrow$ conj.

$\alpha \rightsquigarrow \tilde{\alpha} : BGL(\mathbb{F}_q)^+ \longrightarrow BU$

Quillen's "+" construction.

$\Rightarrow \tilde{\alpha}_* : K_n(\mathbb{F}_q) \longrightarrow \pi_n BU$

Consider $\psi^q - 1 : BU \to BU$

Thm (Quillen) htpy fiber of $\psi^q - 1$ is $BGL(\mathbb{F}_q)^+$.

$$BGL(\mathbb{F}_q)^+ \xrightarrow{\tilde{\alpha}} BU \xrightarrow{\psi^q - 1} BU$$

$$\pi_{2n} BU = \tilde{K}(S^{2n})$$

$\Rightarrow$ $\forall$ finite field $\mathbb{F}_q$, $n \geq 1$

$$K_n(\mathbb{F}_q) = \pi_n(BGL(\mathbb{F}_q)^+)$$

$$\cong \begin{cases} \mathbb{Z}/(q^i - 1), & n = 2i-1 \\ 0, & \text{else.} \end{cases}$$